CATEGORICAL DATA ANALYSIS OF SINGLE SOCIOMETRIC RELATIONS*

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ABSTRACT

Methods based on fitting loglinear models are adapted for the analysis of sociometric relationships among a group of actors, represented as a directed graph. By arranging directed graph data in a four-dimensional cross-classified table, the dyadic relationships between individual actors can be fully studied with a variety of models. These models are based on Holland and Leinhardt's probability density for directed graphs, p_1, but extend their approach to model data from single sociometric generators which include variables measuring nodal attributes. We show how both p_1 and these new models can be fit using standard iterative proportional fitting algorithms. A network of organizations from a midwestern community is used to illustrate these new ideas.

1. Introduction

The use of loglinear models to summarize and describe categorical data in the form of multiple cross-classifications has become increasingly popular in the 1970's. The work of Goodman (e.g., see Goodman 1972) and books by Bishop, Fienberg, and Holland (1975), Fienberg (1977), Haberman (1978), and Upton (1978) have helped to make these methods accessible to researchers with modest statistical background. Because of the flexibility of this new statistical technology, many research problems have been fashioned so that the associated data can be analyzed in the form of a contingency table.

During the same time period, the social network paradigm has also grown in popularity due to increasing evidence that networks can be used to quantify struture in social relationships. We use the phrase "sociometric relation" in the broad sense to refer to any set of sociological connections or associations among a group of social actors or entities. We need not be restricted to Moreno's classical usage of the phrase as interpersonal attitudes of individuals in small, informal, groups. For a group of social actors, one can define many sociometric relations or "generators"; for example, there are three generators in Galaskiewicz's (1979) study of flows of money, information, or support between pairs of organizations in a small Midwestern community. The methods discussed in this chapter are appropriate for studying each of these relations separately. Work is underway on techniques to combine relations and to fit the resulting merged data set with comprehensive models. We discuss multirelational data further in later sections.

A social network is one example of a directed graph, or digraph, a set of g nodes and a set of directed arcs, connecting pairs of nodes. Digraphs are the natural mathematical representations of social networks, and have been used by sociologists since the breakthrough research of Moreno (1934). Many of the mathematical and statistical methods for the analysis of directed graph data have been developed by social scientists. Unfortunately, these methods are rather elementary and rarely make use of contemporary multivariate statistical analysis. (See Sørensen, 1978, for a review of current methodology.). Indeed, a typical analysis of network data makes virtually no use of statistical modelling or inference. In this chapter, we describe some new methods for analyzing social networks based on loglinear models for multivariate categorical data. We draw upon the current interest in both categorical data and social networks, and demonstrate the usefulness of these methods in a simple situation, that of a single sociometric relation.

Our interest in this problem was stimulated through questions raised by Joseph Galaskiewicz, whose data are reported in Table 3 and discussed in Section 5. Several of the loglinear models that we proposed for these data are related to those reported in Galaskiewicz and Marsden (1978). We describe these models and the estimation of their parameters, in Fienberg and Wasserman (1979). These models and others we have fit to Galaskiewicz's data are related to a general class of exponential family models for directed graphs developed by Holland and Leinhardt over the past several years, and reported on by them in a 1979 unpublished manuscript. The models we describe in this chapter are presented in the context of Holland and Leinhardt's model, p_1, using wherever convenient, notation similar to their's.

Holland and Leinhardt (1979) note that there are six basic types of digraphs usually studied:

1) Univariate or single digraphs - single binary relation on a set of nodes.
2) Multivariate digraphs - more than one binary relation on a set of nodes.
3) Univariate digraphs with data on nodal properties or attributes.
4) Multivariate digraphs with data on nodal properties or attributes.
5) Multivalued digraphs with degrees of strengths (non-binary) for one or more relations.
6) Dynamic digraphs, changing over time, for which we have time-series or longitudinal data.

The Holland-Leinhardt model, p_1, is appropriate only for the study of univariate digraphs with no data on nodal attributes, type #1. We have extended p_1 to deal with the complexities of single digraphs with nodal variables, type #3, and discuss these extensions in this chapter. Multivariate digraphs, with or without nodal variables, are briefly discussed in later sections of this chapter and in Fienberg and Wasserman (1979).

On the surface, it might appear that there are few connections between digraphs studied over time, and the categorical data analysis methods used to model types #1-#5. Current methods for analyzing such longitudinal data utilize stochastic models described by Wasserman (1979, 1980) and Runger and Wasserman (1980). By treating the observation of the network at each time point as a new generator, however, these longitudinal data can be viewed as being similar to a multivariate digraph.

The models discussed in this chapter place probability functions on the relations in the group by specifying the probability that a pair of actors has one of four possible dyadic relationships. The entire network of g actors is decomposed into an equivalent set of $\binom{g}{2}$ dyads. To specify the probability distribution of the network, the dyads are assumed to be independent, so that we need merely multiply the dyad probability distributions to obtain their joint distribution. Davis (1968) first proposed the arrangement of data on dyads from one (or two) generator(s) into a 2^2 (or 2^4) dimensional contingency table. The assumption of independent dyads is common to many of the recent models for networks, although it is, at best, an approximation to reality. But building into the models either a dependence structure among the dyads or probability distributions on larger subgraphs such as triads appears very difficult. Davis (1977) worked with triads from single generators structured in the form of 2^6 dimensional table; however, there are many statistical problems in Davis's methodology that must be solved before triadic methods related to this can be correctly applied to the analysis of network data.

Prior to the exposition of our loglinear models for analyzing social network data, we review some necessary notational preliminaries. Following this, we present Holland and Leinhardt's p_1 probability model, and show how this model can be fit using a version of iterative proportional fitting for multidimensional contingency tables. We then discuss several variants on p_1, and extend p_1 to model single relational networks with data on nodal attributes. Throughout this chapter, we analyze various subsets of the corporate interlock data of Galaskiewicz (1979) and Galaskiewicz and Marsden (1978).

2. Some Mathematical Notation

Let D_g be a specific digraph on g nodes from a single generator or binary sociometric relation R . D_g is a binary digraph, with at most one arc connecting node i to node j . Let G denote the set of g nodes. We use the mathematical terms "node" to refer to an individual actor, and "arc" to refer to the presence of a relation between two individual actors. The digraph D_g is described by means of a sociomatrix or adjacency matrix $\underset{\sim}{X}$, with elements (X_{ij}) where

$$X_{ij} = \begin{cases} 1, \text{ if note } i \text{ "chooses" node } j,\ i \rightarrow j. \\ 0, \text{ otherwise.} \end{cases}$$

Note that by convention, we set the g diagonal terms $(X_{ii},\ i = 1,2,\ldots,g)$ to zero.

For a single arc, X_{ij} -- the choice of actor j by actor i -- the arc X_{ji} is the reciprocated choice of actor i by actor j . We usually label

$$D_{ij} = (X_{ij},X_{ji}), \quad i < j,$$

as the dyad, or 2-subgraph, involving the pair of actors i and j . D_{ij} is a bivariate random variable, with $2^2 = 4$ possible realizations. These four realizations and associated labels are:

$D_{ij} = (1,1)$: Mutual,

$D_{ij} = (1,0)$ or $(0,1)$: Asymmetric,

$D_{ij} = (0,0)$: Null.

Also, we define

$$X_{i+} = \sum_j X_{ij}, \quad i = 1,2,\ldots,g$$

$$X_{+j} = \sum_i X_{ij}, \quad j = 1,2,\ldots,g$$

as the outdegree of actor i and indegree of actor j, respectively. The outdegree of a node is the number of arcs emanating from the node, and the indegree of a node is the number of arcs received by the node. A thorough discussion of these and other related network statistics can be found in Harary, Norman, and Cartwright (1965).

When discussing networks as categorical data sets, we generally work with single observations from multiple generators. For a common set of g individuals, and a family of binary relations $R_1, R_2, \ldots, R_n$, we let $\underset{\sim}{X}_r$ be the adjacency matrix for the digraph generated by R_r, with elements (X_{ijr}). The collection of the sociomatrices $\underset{\sim}{X}_1, \underset{\sim}{X}_2, \ldots, \underset{\sim}{X}_n$ is denoted by $\mathfrak{X}$, the multivariate digraph or <u>multigraph</u> of the social system at one point in time. In this chapter, we restrict our attention to a single generator.

3. A Model for Dyadic Interactions in a Single Generator

In Section 2, we described the basic data for the study of dyads with a single generator. Here we introduce an alternative notation. Consider a four dimensional $g \times g \times 2 \times 2$ cross-classification $Y = (Y_{ijk\ell})$, where the subscripts i and j refer to the two actors in a dyad, and k and ℓ refer to the dyad state, so that

$$Y_{ijk\ell} = \begin{cases} 1, \text{ if } D_{ij} = (X_{ij}, X_{ji}) = (k, \ell), \\ 0, \text{ otherwise.} \end{cases}$$

For example, $Y_{ij11} = 1$ if D_{ij} is a mutual dyad. We have abandoned the X-notation in favor of a notation that facilitates analysis of a sociomatrix as a categorical data set. Holland and Leinhardt (1979) prefer to work directly with the X's. The relationship between the X's and Y's can be expressed as follows:

$$\begin{aligned} Y_{ij11} &= X_{ij}X_{ji} \\ Y_{ij10} &= X_{ij}(1-X_{ji}) \\ Y_{ij01} &= (1-X_{ij})X_{ji} \\ Y_{ij00} &= (1-X_{ij})(1-X_{ji}) \quad . \end{aligned}$$

For a given dyad, (i,j), we obtain a 2×2 table of counts, shown in Table 1. Note that $Y_{ij00} + Y_{ij01} + Y_{ij10} + Y_{ij11} = 1$, for all $(i \neq j)$, so that these 2×2 tables contain one 1 and three 0's. Furthermore $Y_{ijk\ell} = Y_{ji\ell k}$, because the dyad (i,j) is the same as the dyad (j,i). We denote a realization of $\underset{\sim}{Y}$ by $\underset{\sim}{y} = (y_{ijk\ell})$. The marginal totals of this 2×2 table, $\underset{\sim}{Y}_{ij}$, correspond to indicator variables for X_{ij} and X_{ji}. Because each of these margins is either (0,1) or (1,0), the interior of the table is completely determined by its marginal totals.

Table 1 goes about here

Let $\mu_{ijk\ell} = \log m_{ijk\ell}$ be elements of a table of log expected values, corresponding to the $Y_{ijk\ell}$. The Holland and Leinhardt density, p_1, for these data begins with the following structure:

$$
(1) \qquad
\begin{aligned}
\mu_{ij00} &= \lambda_{ij} \\
\mu_{ij10} &= \lambda_{ij} + \alpha_i + \beta_j + \theta \\
\mu_{ij01} &= \lambda_{ij} + \alpha_j + \beta_i + \theta \\
\mu_{ij11} &= \lambda_{ij} + \rho_{ij} + \alpha_i + \alpha_j + \beta_i + \beta_j + 2\theta ,
\end{aligned}
$$

subject to the constraints

$$
(2) \qquad e^{\mu_{ij00}} + e^{\mu_{ij10}} + e^{\mu_{ij01}} + e^{\mu_{ij11}} = 1
$$

for all dyads, and

$$
(3) \qquad \sum_{i=1}^{g} \alpha_i = \sum_{j=1}^{g} \beta_j = 0 .
$$

We have reversed Holland-Leinhardt's usage of the $\{\alpha_i\}$ and $\{\beta_j\}$ parameters to correspond to the standard ANOVA notation. The parameters $\{\alpha_i\}$ measure the "expansiveness" or the "productivity" of the actors, reflecting how likely an actor is to "produce" new relational ties. The parameters $\{\beta_j\}$ measure the "popularity" or the "attractiveness" of the actors, reflecting how likely an actor is to "attract" new relational ties. The $\{\rho_{ij}\}$ parameters are "reciprocity" measures, and specify how likely

Table 1

2×2 Dyadic Contingency Table

		Actor j → Actor i: No	Actor j → Actor i: Yes	
Actor i → Actor j	No	Y_{ij00}	Y_{ij01}	$1-X_{ij}$
	Yes	Y_{ij10}	Y_{ij11}	X_{ij}
		$1-X_{ji}$	X_{ji}	1

it is that $i \to j$ if $j \to i$; i.e., the increase in the probability of a relational tie forming between two actors, if the reciprocated tie is present. The parameters $\{\lambda_{ij}\}$ are "dyadic" effects, and are present in (1) to assure that the sampling constraints (2) hold. (Holland and Leinhardt use a slightly different notation for these normalizing constants.)

The model defined by equations (1) has many parameters, so many that we can not get separate estimates of $\{\rho_{ij}\}$ and $\{\lambda_{ij}\}$, the reciprocity and dyad parameters. Consequently, maximum likelihood estimation of the cells in this table not only causes an identification problem, but leads to degeneracies in the table of fitted values, and the model can be seen to fit the data $\underset{\sim}{y}$ perfectly.

3.1 The p_1 exponential family of densities

As mentioned, the model specified by equations (1) with constraints (2) and (3) cannot be fitted to a sociomatrix because of under-identification of the parameters. One solution to this problem is to simplify the model so that the reciprocity effects $\{\rho_{ij}\}$ are constant across all dyads; i.e., we revise model (1) so that

$$\rho_{ij} \equiv \rho \qquad \text{for all } i \neq j \quad . \tag{4}$$

Equation (4), coupled with equations (1), (2), and (3) is termed the p_1-density by Holland and Leinhardt (1979).

Assuming that the dyads $\{D_{ij}\}$ are statistically independent, then the log likelihood function for p_1, is

$$\log L(\{\alpha_i\},\{\beta_j\},\{\lambda_{ij}\},\rho,\theta|\underset{\sim}{y})$$

$$= \log P\{\underset{\sim}{Y}=\underset{\sim}{y}\}$$

$$= \rho \sum_{i<j} y_{ij11} + \sum_i \alpha_i (y_{i+10} + y_{i+11})$$

(5)

$$+ \sum_j \beta_j (y_{+j10} + y_{+j11}) + \theta (y_{++10} + y_{++11}) + \sum_{i<j} \lambda_{ij}$$

$$= \frac{\rho}{2} y_{++11} + \sum_i \alpha_i y_{i+1+} + \sum_j \beta_j y_{+j1+} + \theta y_{++1+} + \sum_{i<j} \lambda_{ij}$$

subject to the constraints (2) that the $m_{ijk\ell}$ sum to one for each dyad. Note, that in terms of the original sociomatrix $\underset{\sim}{X}$, the sufficient statistics for the parameters of p_1 are:

(6a)	$\frac{1}{2} y_{++11} = \sum_{i<j} x_{ij} x_{ji}$	Number of Mutuals
(6b)	$(y_{i+1+}) = x_{i+}$	Outdegree of node i $i = 1,2,\ldots,g$
(6c)	$(y_{+j1+}) = x_{+j}$	Indegree of node j $j = 1,2,\ldots,g$
(6d)	$(y_{++1+}) = x_{++}$	Total number of choices.

Therefore, fitting the p_1 model to an "observed" sociomatrix is equivalent to constructing an "expected" sociomatrix with indegrees, outdegrees, number of mutuals, and total number of choices identical to those of the observed sociomatrix. We then ask how much the expected and observed sociomatrices differ. A large difference is evidence that the group exhibits structural properties that are not due to simply the sufficient statistics; i.e., to model such a group, one needs a more sophisticated model incorporating parameters for additional structural effects, such as differential mutuality, choices made only within subgroups or "cliques", etc. We discuss several alternative models in later sections of this chapter.

3.2 Fitting p_1 to data

To fit the p_1 density function to data, we can differentiate the log likelihood function (5), to obtain a series of equations whose solutions gives the maximum likelihood estimates, $\hat{m}_{ijk\ell}$, of the elements of the $\underset{\sim}{Y}$ array. These equations are

(7a) (ρ-step)

$$\hat{m}_{++11} = Y_{++11}$$

(7b) (row-step)

$$\begin{cases} \hat{m}_{i+10} + \hat{m}_{i+11} = Y_{i+10} + Y_{i+11}, \\ \quad i = 1,\ldots,g \\ \hat{m}_{j+10} + \hat{m}_{j+11} = Y_{j+10} + Y_{j+11}, \\ \quad j = 1,\ldots,g \end{cases}$$

(7c) (column-step)

$$\begin{cases} \hat{m}_{+i10} + \hat{m}_{+i11} = Y_{+i10} + Y_{+i11}, \\ \quad i = 1,\ldots,g \\ \hat{m}_{+j10} + \hat{m}_{+j11} = Y_{+j10} + Y_{+j11}. \\ \quad j = 1,\ldots,g \end{cases}$$

(7d) (Normalizing)

$$\hat{m}_{ij00} + \hat{m}_{ij10} + \hat{m}_{ij01} + \hat{m}_{ij11} = 1, \; i < j.$$

This set of equations basically set margins of the $\underset{\sim}{\hat{m}}$ array equal to comparable margins of the $\underset{\sim}{Y}$ array. Following Darroch and Ratcliff (1972), one can then specify an algorithm to fit p_1 by iteratively adjusting the elements of the $\underset{\sim}{\hat{m}}$ table to have the desired margins. Because of the symmetrics in the $\underset{\sim}{Y}$ array, two sets of equations must be solved for the $\{\alpha_i\}$, equations (7b), and two sets for the $\{\beta_j\}$, equations (7c). The adjusted fitted values eventually converge to the maximum likelihood estimates, but only after many iterations.

It turns out, however, that fitting p_1 to either the $\underset{\sim}{X}$-array or $\underset{\sim}{Y}$-array is equivalent to fitting the "no three-factor interaction" loglinear model to $\underset{\sim}{y}$. Thus, one can use a standard contingency table, iterative proportional fitting computer program, such as BMDP3F (Dixon and Brown, 1979, chapter 11.3), and one need not do any FORTRAN (or other computer language) programming to fit p_1. This fact should make the analysis of networks as categorical data sets much easier for the researcher.

A formal proof of the equivalence of the iterative scaling algorithm for p_1 and the standard iterative scaling algorithm for the no three-factor interaction loglinear model for $\underset{\sim}{Y}$ is rather complicated and is not included here. The no three-factor interaction loglinear model in the notation of Fienberg (1977), fits the following margins to the $\hat{\underset{\sim}{m}}$ array:

$$(8) \qquad [12] \quad [13] \quad [24] \quad [14] \quad [23] \quad [34] \quad .$$

The effects associated with the [12]-margin $(\hat{m}_{ij++})$ are related to the $\{\lambda_{ij}\}$ parameters; the effects associated with the [13]- and [24]-margins $(\hat{m}_{i+k+}$ and $\hat{m}_{+i+\ell}$, which are equal) are identical to the $\{\alpha_i\}$ parameters; those for the [14]- and [23]-margins $(\hat{m}_{i++\ell}$ and $\hat{m}_{+jk+}$, also equal) are identical to the $\{\beta_j\}$ parameters; and lastly, the remaining degree of freedom associated with the [34]-margin $(\hat{m}_{++k\ell})$ corresponds to ρ. The equation for θ is redundant given the g equations for either the $\{\alpha_i\}$, equations (7b), or the $\{\beta_j\}$, equations (7c).

The expected values for the elements of the sociomatrix $\underset{\sim}{X}$ are then

$$(9) \qquad \hat{x}_{ij} = \hat{y}_{ij1+} = \hat{m}_{ij1+} = \hat{m}_{ij10} + \hat{m}_{ij11} \ .$$

Determining the number of degrees of freedom associated with p_1 is somewhat complex. Consider each 2×2 table, containing one 1 and three zeros. These 2×2 tables are determined by their margins since knowledge of the margins allows the tables to be filled in. Since these one-dimensional margins are constrained to sum to unity, they each have only one degree of freedom. Therefore, there are 2 degrees of freedom for each 2×2 table, and hence, g(g-1) degrees of freedom in $\underset{\sim}{Y}$, exactly the number of degrees of freedom in $\underset{\sim}{X}$. Also note that these one-dimensional margins not only determine the interior of the 2×2 tables, but also determine the entries in $\underset{\sim}{X}$, since

$$X_{ij} = Y_{ij10} + Y_{ij11} = Y_{ij1+}$$

$$X_{ji} = Y_{ij01} + Y_{ij11} = Y_{ij+1} .$$

Finally, we lose 1 degree of freedom each for θ and ρ , and (g-1) degrees of freedom each for $\{\alpha_i\}$ and $\{\beta_j\}$ which leaves $g(g-1)-1-1-(g-1)-(g-1) = g(g-3)$ degrees of freedom for a goodness-of-fit test of p_1 .

If we treat these dyadic interactions as a stochastic process, and have multiple observations on $\underset{\sim}{X}$, then we can form a $\underset{\sim}{Y}$ array that contains the frequencies with which each dyad type occurs. With n observations, each 2×2 table sums to n . In this situation, $\underset{\sim}{Y}$ has its full complement of $4\binom{g}{2}$ degrees of freedom. But, with only one observation on $\underset{\sim}{X}$, $\underset{\sim}{Y}$ has only g(g-1) degrees of freedom. One can now see why the model given by equations (1) cannot be estimated. The $\{\rho_{ij}\}$ have $\binom{g}{2}$ degrees of freedom and so do the $\{\lambda_{ij}\}$ leaving no degrees of freedom with which to estimate the remaining parameters.

We fit p_1 to data in section 5 of this chapter.

3.3 Estimation of the p_1 parameters

We now give maximum likelihood estimates of the parameters of the p_1 density function: $\{\alpha_i\}$, $\{\beta_j\}$, ρ, and θ . The p_1 density function is specified by equations (1) and (4), with constraints (2) and (3). By taking ratios of elements of $\underset{\sim}{m}$, we find the following identities involving the model parameters:

$$(10) \qquad \rho = \log\left(\frac{m_{ij00}\ m_{ij11}}{m_{ij10}\ m_{ij01}}\right), \quad i < j .$$

$$(11) \qquad \theta + \beta_i + \alpha_j = \log\left(\frac{m_{ij10}}{m_{ij00}}\right), \quad i \neq j .$$

To compute $\hat{\rho}$, we use the fact, given by equation (10), that the parameter is the logarithm of the cross-product ratio of each 2×2 table; i.e.

$$\log \frac{\hat{m}_{ij00}\ \hat{m}_{ij11}}{\hat{m}_{ij10}\ \hat{m}_{ij01}}$$

for every dyad (i,j). For numerical stability of our estimate we average these quantities yielding

$$(12) \qquad \hat{\rho} = \frac{2}{g(g-1)} \sum_{i<j} \log\left(\frac{\hat{m}_{ij00}\ \hat{m}_{ij11}}{\hat{m}_{ij10}\ \hat{m}_{ij01}}\right)$$

Next, the maximum likelihood estimates $\{\alpha_i\}$ can be computed by noting, from equation (11), that

$$(13) \qquad \hat{\alpha}_i - \hat{\alpha}_{i'} = \log\left(\frac{\hat{m}_{ij10}/\hat{m}_{ij00}}{\hat{m}_{i'j10}/\hat{m}_{i'j00}}\right), \quad i \neq i'$$

and that $\Sigma\alpha_i = 0$. Similarly, the $\{\hat{\beta}_j\}$ estimates obey the equations

$$(14) \qquad \hat{\beta}_j - \hat{\beta}_{j'} = \log\left(\frac{\hat{m}_{ij10}/\hat{m}_{ij00}}{\hat{m}_{ij'10}/\hat{m}_{ij'00}}\right), \quad j \neq j'$$

with $\Sigma\beta_j = 0$. Lastly, given the $\{\hat{\alpha}_i\}$ and $\{\hat{\beta}_j\}$ parameters, we can compute $\hat{\theta}$ by using (11) averaged overall (i,j):

$$(15) \qquad \hat{\theta} = \frac{1}{g(g-1)} \sum_{i \neq j} \left[\log\left(\frac{\hat{m}_{ij10}}{\hat{m}_{ij00}}\right) - \hat{\alpha}_i - \hat{\beta}_j\right] .$$

4. Extensions of p_1 for Single Sociometric Relations

The p_1 density is a basic model for data on single sociometric relations without nodal attributes. It has parameters for attractiveness $(\{\beta_j\})$, expansiveness $(\{\alpha_i\})$, and reciprocity (ρ). A lack of fit of p_1 to a specific data set indicates that the group under investigation exhibits additional structural properties. In this section we briefly discuss several alternatives and specializations of p_1, and suggest some tests of hypothesis for the parameters of p_1.

One can derive special cases of p_1 by setting various parameters of p_1 to zero. Most of these special cases are outlined by Holland and Leinhardt (1979). We list these special cases and others in Table 2, and, in addition, specify the loglinear model for Y "equivalent" to each special case. Of these special cases, (i), (ii), (iii), (iv), and (vii) are most important. Case (ii), which we label $p_{.75}$, postulates that $\rho=0$. Case (iii), which is unlabelled, postulates that the popularity parameters $\{\beta_j\} = 0$, and (iv) combines (ii) and (iii). Case (vii), labelled $p_{.5}$, stipulates that both $\{\alpha_i\} = \{\beta_j\} = 0$. In most sociometric studies, one wants a model with the expansiveness parameters $\{\alpha_i\}$. Outdegrees are very often fixed by the design of the sociometric study; thus, one might always choose to fit the [13] and [24] margins to $\underset{\sim}{y}$ and thus not consider cases (v) through (viii).

The last model given in Table 2, labelled $p_{1.5}$, is a model for differential mutuality. It cannot be obtained by setting p_1 parameters to zero. Rather, we take the full model specified by equations (1), and further postulate that

$$(16) \qquad \rho_{ij} = \rho + \rho_{1(i)} + \rho_{1(j)}, \quad i \neq j;$$

Table 2. Models for Single Relations Based on p_1.

Special Case	Label	Parameters	Margins for Loglinear model	degrees of freedom
(i)	p_1	$\rho, \theta, \{\alpha_i\}, \{\beta_j\}$	[12] [13] [14] [23] [24] [34]	g(g-3)
(ii)	$p_{.75}$	$\theta, \{\alpha_i\}, \{\beta_j\}$	[12] [13] [14] [23] [24]	g(g-3)+1
(iii)		$\rho, \theta, \{\alpha_i\}$	[12] [13] [24] [34]	g(g-2)-1
(iv)		$\theta, \{\alpha_i\}$	[12] [13] [24]	g(g-2)
(v)		$\rho, \theta, \{\beta_j\}$	[12] [14] [23] [34]	g(g-2)-1
(vi)		$\theta, \{\beta_j\}$	[12] [14] [23]	g(g-2)
(vii)	$p_{.5}$	ρ, θ	[12] [34]	(g+1)(g-2)
(viii)		θ	[12] [3] [4]	g(g-1)-1
(ix)	$p_{1.5}$	$\rho, \{\rho_{1(i)}\}, \theta, \{\alpha_i\}, \{\beta_j\}$	[12] [134] [234]	g(g-4)+1

where the $\rho_{1(i)}$ are normalized to sum to zero, i.e., the mutuality parameters depend in a linear fashion on individuals i and j . The $\{\rho_{1(i)}\}$ parameters measure that rates at which individuals are likely to enter into mutual, symmetric relationships. Fitting this $p_{1.5}$ density function is equivalent to fitting a loglinear model to $\underset{\sim}{Y}$ which, in addition to the dyad, [12]-margin, fits two three-factor interactions corresponding to [134] and [234]. We work with these models in the next section.

To test whether the parameters of p_1 are nonzero, consider the hypotheses:

$$H_1: \ \rho=0,$$
$$H_2: \ \alpha_1=\alpha_2=\ldots=\alpha_g=0,$$
$$H_3: \ \beta_1=\beta_2=\ldots=\beta_g=0,$$
$$H_4: \ \rho_{1(1)}=\rho_{1(2)}=\ldots=\rho_{1(g)}=0.$$

By computing likelihood ratio statistics comparing pairs of models in Table 2, one can obtain test statistics for these four hypotheses. For example, the likelihood ratio statistic for comparing cases (ii) and (i), $G^2[(ii)|(i)]$ in the notation of Fienberg (1977), can be used to test H_1 , but this test is conditional on the p_1 density being appropriate for the data. For the four hypotheses, we suggest computing the likelihood ratio statistics comparing the following pairs of models:

Hypotheses	Pairs of Models to be Compared
H_1	(ii) and (i)
H_2	(v) and (i)
H_3	(iii) and (i)
H_4	(i) and (ix)

(Note that the likelihood ratio for comparing models (viii) and (vii) also corresponds to a test of H_1, but it assumes that the $p_{.5}$ density is appropriate for the data. This is far more restrictive than assuming the appropriateness of p_1.)

The asymptotic distributions for these test statistics are difficult to derive, but there is strong evidence (Holland and Leinhardt, 1979) that the distributions are χ^2, with 1 degree of freedom for H_1, and (g-1) degrees of freedom for H_2, H_3, and H_4.

5. An Example

We demonstrate the use of p_1 and its relatives on a data set discussed by Galaskiewicz and Marsden (1978). They study the interrelationships among 73 out of a total of 109 organizations in a Midwest community with 32,000 residents, in terms of their pairwise or dyadic relationships. These organizations are described in more detail in Galaskiewicz (1979). If we denote by A and B the two organizations in a pair, then the six relationships analyzed by Galaskiewicz and Marsden are (i) and (ii), the flow of money from A to B and from B to A, (iii) and (iv), the flow of information from A to B and from B to A and finally (v) and (vi), the flows of "support". Following Davis (1977), they cross-classify the $\binom{73}{2}$ dyadic relationships according to these six binary variables, and proceed to fit standard loglinear models. Table 3 contains the data of Galaskiewicz and Marsden, and Figure 1 contains 5 possible patterns of flow for a "typical" dyad which they attempt to incorporate into their models. We discuss models and methods of analysis for these aggregate data in Table 3 in Fienberg and Wasserman (1979).

Throughout the remainder of this chapter, we will study a disaggregated version of these data in the form of three 73×73 sociomatrices for the three media, money, information and support. We analyze the three relations separately here, and defer to a later paper a description of more complex analyses linking them together.

The p_1 density function is difficult to fit to a large group. Not only must one construct a $g \times g \times 2 \times 2$ table, which can contain many cells when g is large, but one must also deal with an overabundance of parameters, making the comprehension of the fit and parameters rather difficult. One solution to this problem is to group the actors into subgroups, and equate the parameters for all actors in a subgroup. This approach is discussed in the next section.

Table 3

Observed Distribution of Interorganizational Transactions Involving Three Resources and 73 Organizations[a] (Source: Galaskiewicz and Marsden (1978))

Information out		–								+							
Information in		–				+				–				+			
Money out		–		+		–		+		–		+		–		+	
Money in		–	+	–	+	–	+	–	+	–	+	–	+	–	+	–	+
Support out	Support in																
–	–	3020	89	89	24	145	16	17	8	145	17	16	8	332	47	47	16
	+	115	17	11	3	21	9	4	4	31	18	2	1	77	37	16	25
+	–	115	11	17	3	31	2	18	1	21	4	9	4	77	16	37	25
	+	110	13	13	4	19	4	7	0	19	7	4	0	102	52	52	32

Table total = 5256

[a]"+" indicates that a directed flow is present, "–" indicates that a directed flow is absent.

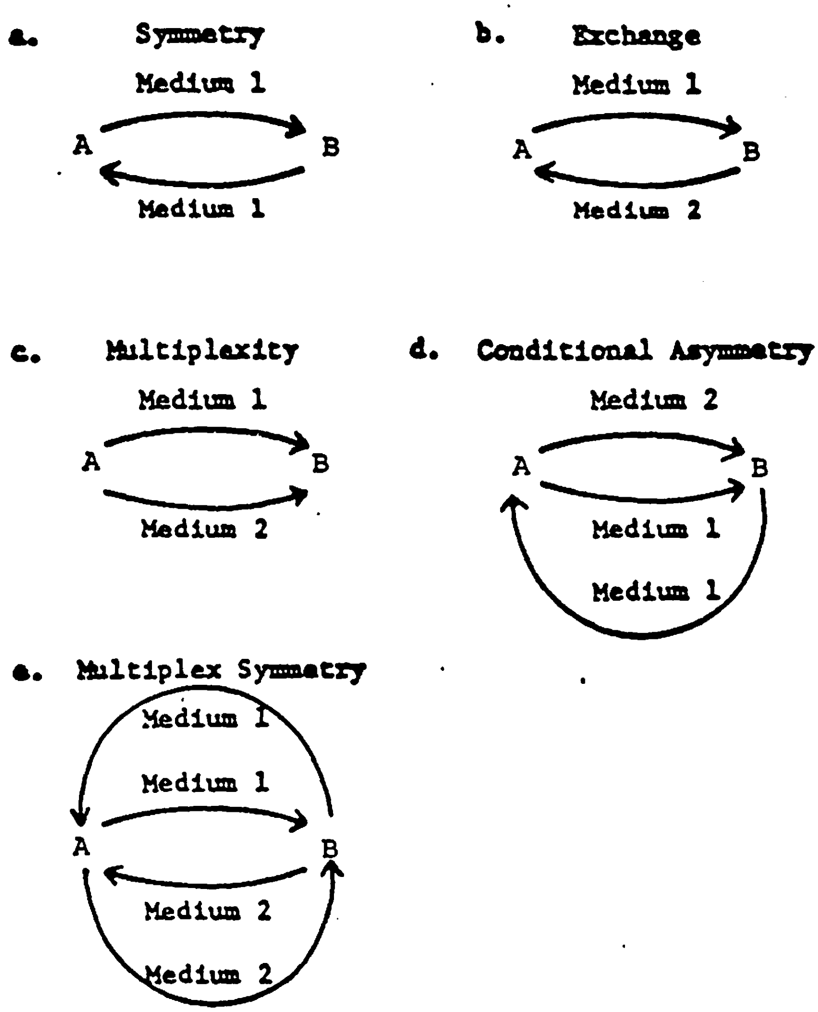

Figure 1. Patterns of flow dependency.
Source: Galaskiewicz and Marsden (1978)

To illustrate the models discussed in sections 3 and 4, we take only a subset of the 73 organizations. Of the 73, 16 are business organizations and we study the information flows among these 16 organizations. Two questions on information flows were posed to the organizations: A) Which organizations in the community does your organization rely upon for information regarding community affairs; and B) To which organizations in the community would your organization be likely to pass on important information concerning community affairs? From these lists we get Table 4 in which a "1", denoting the presence of a flow of information from i to j is recorded if either organization i listed organization j on question B, or j listed i on question A. The pseudonyms for the organizations are from Galaskiewicz (1979, page 47). Note the large amount of symmetry present and the near equivalence of the indegrees and outdegrees; 35.8% of the 120 dyads are mutual and only 5.0% are asymmetric.

We fit the p_1 density to these data using the approach discussed in the previous section. This yields the expected values of the entries in the sociomatrix in Table 5. (Recall that $E(X_{ij}) = m_{ij10} + m_{ij11}$.) For these data, the likelihood ratio statistic for the fit of the p_1 density is

$$(17) \qquad G^2 = 2 \sum_{i<j} \sum_{k,\ell} y_{ijk\ell} \log \frac{y_{ijk\ell}}{\hat{m}_{ijk\ell}} = 104.8$$

with 208 degrees of freedom. A stem-and-leaf display of the residuals from the model $x_{ij} - \hat{m}_{ij1+}$, is shown in Figure 2. Because of the present lack of asymptotic theory, we do not know of a good way to norm the residuals in order to study their individual magnitudes.

Table 4

Sociomatrix of Business Organizations based on Information Flows from Galaskiewicz (1979) and Associated Statistics.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	Outdegree
1. Farm Equipment Co.		1	1	1	1	1	1	0	0	0	0	0	0	0	0	1	7
2. Clothing Mfg. Co.	1		0	0	0	0	1	0	0	0	0	0	0	0	0	1	3
3. Farm Supply Co.	1	0		0	0	0	0	1	0	1	0	1	0	1	0	1	6
4. Mechanical Co.	1	0	0		0	0	0	0	0	0	0	0	0	0	0	1	2
5. Electrical Equip. Co.	1	0	0	0		1	0	1	0	0	0	0	0	0	1	1	5
6. Metal Products Co.	1	0	0	0	1		0	1	0	0	0	0	0	0	0	1	4
7. Music Equipment Co.	1	1	0	0	0	0		1	0	0	0	0	0	0	0	1	4
8. 1st Towertown Bank	0	0	1	0	1	1	1		0	1	0	1	0	1	1	1	9
9. Towertown Svgs. and Loan	0	0	0	0	0	1	0	1		1	0	1	1	1	1	1	8
10. Bank of Towertown	0	0	1	0	0	0	0	1	0		0	1	0	1	1	1	6
11. 2nd Towertown Bank	0	0	0	0	0	0	0	0	0	0		0	0	0	0	1	1
12. Brinkman Law Firm	0	0	1	0	0	1	0	1	1	1	0		1	1	0	0	7
13. Cater Law Firm	0	0	0	0	0	1	0	1	1	0	0	1		0	0	0	4
14. Knapp Law Firm	0	0	1	0	0	0	0	1	1	1	0	1	0		1	1	7
15. Towertown News	0	0	0	0	1	0	0	1	1	1	0	0	0	1		1	6
16. WTWR Radio	1	1	1	1	1	1	1	1	1	1	1	0	0	1	1		13
Indegree	7	3	6	2	5	7	4	11	5	7	1	6	2	7	6	13	92

$\binom{16}{2}$ = 120 dyads; of these, 43 are mutuals, 6 are asymmetric, and 71 are null.

Table 5

Expected Values of Information Flows between Galaskiewicz's

Business Organizations

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1. Farm Equipment Co.		220	531	128	426	420	321	859	596	582	052	573	170	629	531	960
2. Clothing Mfg. Co.	220		159	024	110	114	073	504	202	189	009	183	033	220	159	801
3. Farm Supply Co.	531	159		090	332	328	240	804	499	483	036	473	121	531	432	942
4. Mechanical Co.	128	024	090		061	065	039	347	117	108	005	104	017	128	090	677
5. Electrical Equip. Co.	426	110	332	061		245	172	728	397	380	024	371	082	426	332	914
6. Metal Products Co.	406	101	309	056	228		158	711	0	360	022	050	001	400	309	900
7. Music Equipment Co.	321	073	240	039	172	174		630	296	280	015	272	054	321	240	871
8. 1st Towertown Bank	858	501	801	344	726	711	627		008	833	164	724	059	858	801	988
9. Towertown Svgs. and Loan	616	215	518	127	415	1000	312	998		934	054	552	159	616	518	958
10. Bank of Towertown	581	188	482	108	378	362	279	833	089		043	520	123	581	482	952
11. 2nd Towertown Bank	052	009	036	005	024	028	015	166	047	043		042	007	052	036	440
12. Brinkman Law Firm	574	184	475	105	372	902	274	852	551	529	042		140	574	475	951
13. Cater Law Firm	171	033	121	018	083	990	054	918	158	261	007	140		171	121	747
14. Knapp Law Firm	629	220	531	128	426	420	321	859	596	582	052	573	170		531	960
15. Towertown News	531	159	432	090	332	328	240	804	499	483	036	473	121	531		942
16. WTWR Radio	960	801	942	677	914	909	871	988	949	952	440	950	744	960	942	

Entries are $\times 10^{-3}$

Figure 2

Stem-and Leaf Display of Residuals (differences between) observed and fitted values) for Galaskiewicz's Data

COUNT	PERCENT		
1	.4	-9	5
4	1.6	-8	7440
1	.4	-7	8
4	1.6	-6	4440
8	3.3	-5	98443300
10	4.2	-4	9885333330
17	7.1	-3	98877666543322221
18	7.5	-2	999877553333311110
31	12.9	-1	9998777666666666555544221110000
52	21.7	-0	9999888887555555444444433333333333322222221111100000
18	7.6	0	013333334445556677
14	5.8	1	00022345556677
10	4.2	2	1134666889
7	2.9	3	1144689
17	7.1	4	01111555555566669
12	5.1	5	000112233666
6	2.5	6	266778
2	.8	7	88
6	2.5	8	555678
2	.8	9	33
240	100		

If we round the expected values in Table 5 to either 0 (if entry is <.50) or 1 (if entry is >.50) we can compare the observed and expected values in an easily interpretable manner. The rounded table of expected values is shown in Table 6, along with information on the discrepancies of this table with the observed data. A single entry indicates agreement between the rounded expected value and the observed; double entries indicate a discrepancy; e.g., if the residual for the cell is <-.50, the entry in Table 6 is 0/1. We can see that of the 240 entries, there are 45 "errors", 28 1/0's and 17 0/1's. Moreover, 22 of the 45 involve actor #1, a Farm Equipment Company. Actor #1 sends and receives information from corporate actors #2, 3, 4, 5, 6, 7, 16; however, Table 6 indicates that this firm should send and receive information from actors #3, 8, 9, 10, 12, 14, 15, 16. Thus, the Farm Equipment Company behaves contradictorily to the group as a whole. This behavior may be because the company is not locally owned and is located on the outskirts of the community. The company simply may have no need for information from the financial or legal resources in the community.

The parameter estimates for the group of 16 business organizations are given at the left in Table 7. We can note the large reciprocity effect ($\hat{\rho}$ = 29.80), implying a large chance that a dyad chosen at random is a mutual. Actors #9, 12, 13, and 16 are likely, and #6, 8, and 10 unlikely, to send information to other actors. Actors #6, 8, 10, and 16 are likely, and #9, 12, and 13 unlikely, to receive information from other actors.

Since the Farm Equipment Company sends and receives information from an unexpected set of firms, we ignore this first actor and fit the model

Table 6

Rounded Table of Expected Values for Galaskiewicz Data

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1. Farm Equipment Co.		1/0	1	1/0	1/0	1/0	1/0	0/1	0/1	0/1	0	0/1	0	0/1	0/1	1
2. Clothing Mfg. Co.	1/0		0	0	0	0	1/0	0/1	0	0	0	0	0	0	0	1
3. Farm Supply Co.	1	0		0	0	0	0	1	0	1/0	0	1/0	0	1	0	1
4. Mechanical Co.	1/0	0	0		0	0	0	0	0	0	0	0	0	0	0	1
5. Electrical Equip. Co.	1/0	0	0	0		1/0	0	1	0	0	0	0	0	0	1/0	1
6. Metal Products Co.	1/0	0	0	0	1/0		0	1	0	0	0	0	0	0	0	1
7. Music Equipment Co.	1/0	1/0	0	0	0	0		1	0	0	0	0	0	0	0	1
8. 1st Towertown Bank	0/1	0/1	1	0	1	1	1		0	1	0	1	0	1	1	1
9. Towertown Svgs. and Loan	0/1	0	0/1	0	0	1	0	1		1	0	1	1/0	1	1	1
10. Bank of Towertown	0/1	0	1/0	0	0	0	0	1	0		0	1	0	1	1/0	1
11. 2nd Towertown Bank	0	0	0	0	0	0	0	0	0	0		0	0	0	0	1/0
12. Brinkman Law Firm	0/1	0	1/0	0	0	1	0	1	1	1	0		1/0	1	0	1
13. Cater Law Firm	0	0	0	0	0	1	0	1	1/0	0	0	1/0		0	0	0/1
14. Knapp Law Firm	0/1	0	1	0	0	0	0	1	1	1	0	1	0		1	1
15. Towertown News	0/1	0	0	0	1/0	0	0	1	1/0	0	0	0	0	1		1
16. WTWR Radio	1	1	1	1	1	1	1	1	1	1	1/0	1	0/1	1	1	

Note: If rounded expected value equals observed value only one entry is given; otherwise, entry is of form observed value/rounded expected value.

to the remaining 15. The matrix of rounded expected values is shown in Table 8. There are now 27 errors, 16 1/0's and 11 0/1's, but they are now scattered at random throughout the table and 19 of these are in similar positions as the errors in Table 5. The likelihood ratio statistic for this revised model is $G^2 = 72.7$, with 180 degrees of freedom.

The parameter estimates for the reduced set of 15 firms omitting #1, are given in Table 7. Reciprocity is still large ($\hat{\rho} = 29.76$) only slightly less than in the larger data set. Actors #9, 12, 13, and 16 are still likely to send information, but now actors #6, 8, 10, and 4, and 11 are unlikely to do so. Actors #6, 8, 10, and 16 are still likely to receive information, but now, in addition to actors #9, 12, and 13, actors #4, and 11 are unlikely to receive any. By ignoring actor #1, we have increased the chance that most actors send or receive information. Only actors #2, 3, 5, 7, 14, and 15 are "neutral", being likely to neither send nor receive information. Without the outlying actor #1, we have a more "integrated" network, with 9 actors playing essential roles in soliciting and distributing information.

In addition to p_1, we fit other models to the $\underset{\sim}{y}$ array for the 16 actors. The likelihood ratio statistics, degrees of freedom, and number of iterations required for the fitted values to converge to the maximum likelihood estimates for these models are given in Table 9. Note that both p_1 and $p_{1.5}$ required many, many iterations, while the other four models converged rapidly.

Hypothesis tests for the p_1 parameters are given at the bottom of Table 9. The test statistics for the four hypotheses are differences of the likelihood ratio statistics; for example, to test H_2, we subtract the G^2 for p_1, $G^2_{(1)}$, from the G^2 for the sixth model $p_{.75}$, which we call $G^2_{(6)}$, to obtain 219.97-104.76=115.21 (see Fienberg, 1977, for further details on such conditional

Table 7

Parameter Estimates for Galaskiewicz's Data

	All Actors	Without actor #1		All actors	Without actor #1
$\hat{\alpha}_1$	.228		$\hat{\beta}_1$	.328	
$\hat{\alpha}_2$	-.455	-.582	$\hat{\beta}_2$	-.557	-.557
$\hat{\alpha}_3$	.081	.101	$\hat{\beta}_3$	.123	.050
$\hat{\alpha}_4$	-.725	-1.053	$\hat{\beta}_4$	-.857	-1.608
$\hat{\alpha}_5$	-.081	.064	$\hat{\beta}_5$	-.089	-.396
$\hat{\alpha}_6$	-12.670	-13.019	$\hat{\beta}_6$	12.436	12.411
$\hat{\alpha}_7$	-.249	-.278	$\hat{\beta}_7$	-.312	-.738
$\hat{\alpha}_8$	-8.290	-8.179	$\hat{\beta}_8$	9.961	10.812
$\hat{\alpha}_9$	12.702	13.121	$\hat{\beta}_9$	-12.224	-12.074
$\hat{\alpha}_{10}$	-5.636	-5.486	$\hat{\beta}_{10}$	6.019	6.368
$\hat{\alpha}_{11}$	-1.143	-1.053	$\hat{\beta}_{11}$	-1.300	-1.608
$\hat{\alpha}_{12}$	6.013	6.218	$\hat{\beta}_{12}$	-5.659	-5.484
$\hat{\alpha}_{13}$	8.632	8.373	$\hat{\beta}_{13}$	-9.922	-9.618
$\hat{\alpha}_{14}$	.228	.371	$\hat{\beta}_{14}$	.328	.650
$\hat{\alpha}_{15}$	.081	.243	$\hat{\beta}_{15}$	.123	.311
$\hat{\alpha}_{16}$	1.278	1.288	$\hat{\beta}_{16}$	1.602	2.137

	All actors	Without actor #1
$\hat{\theta}$	-15.230	-15.360
$\hat{\rho}$	29.800	29.764

Table 8

Rounded Table of Expected Values for Galaskiewicz's

data without Actor #1

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1. Farm Equipment Co.																
2. Clothing Mfg. Co.			0	0	0	0	1/0	0/1	0	0	0	0	0	0	0	1
3. Farm Supply Co.		0		0	0	0	0	1	0/1	1	0	1/0	0	1	0	1
4. Mechanical Co.		0	0		0	0	0	0	0	0	0	0	0	0	0	1/0
5. Electrical Equip. Co.		0	0	0		1/0	0	1	0	0	0	0	0	0	1/0	1
6. Metal Products Co.		0	0	0	1/0		0	1	0	0	0	0	0	0	0	1
7. Music Equipment Co.		1/0	0	0	0	0		1	0	0	0	0	0	0	0	1
8. 1st Towertown Bank		0/1	1	0	1	1	1		0	1	0	1	0	1	1	1
9. Towertown Svgs. and Loan		0	0/1	0	0	1	0	1		1	0	1	1/0	1	1	1
10. Bank of Towertown		0	1/0	0	0	0	0	1	0		0	1	0	1	1	1
11. 2nd Towertown Bank		0	0	0	0	0	0	0	0	0		0	0	0	0	1/0
12. Brinkman Law Firm		0	1	0	0	1	0	1	1	1	0		1/0	1	0/1	0/1
13. Cater Law Firm		0	0	0	0	1	0	1	1/0	0	0	1/0		0	0	0/1
14. Knapp Law Firm		0	1	0	0	0/1	0	1	1	1	0	1	0		1	1
15. Towertown News		0	0	0	1/0	0	0	1	1	1	0	0/1	0	1		1
16. WTWR Radio		1	1	1/0	1	1	1	1	1	1	1/0	0/1	0/1	1	1	

Entries are defined in footnote to Table 6.

tests). We see that there is little evidence that any of the parameters in the p_1 density model are zero, and that there is strong evidence that the group of organizations does not exhibit differential mutuality. The conclusion that the $\{\rho_1\}$ are zero may be due to the lack of asymmetric choices in the network. Since virtually all choices are symmetric, the $\{\rho_1\}$ are not needed, given that the $\{\alpha\}$ and $\{\beta\}$ parameters are present in the model.

Table 9. Models fit to Galaskiewicz's 16 Business Organizations, and Related Tests of Hypotheses

	Model	Margins Fitted to y	Parameters	G^2	df	No. of Iterations
1)	p_1	[12] [13] [14] [23] [24] [34]	$\theta\{\alpha\}\{\beta\}\rho$	104.76	208	350
2)	$p_{.5}$	[12] [34]	$\theta\ \rho$	207.05	238	1
3)	$p_{1.5}$	[12] [134] [234]	$\theta\{\alpha\}\{\beta\}$ $\rho\{\rho_1\}$	102.00	193	350
4)		[12] [13] [24] [34]	$\theta\{\alpha\}\rho$	163.13	223	6
5)		[12] [14] [23] [34]	$\theta\{\beta\}\rho$	154.67	223	7
6)	$p_{.75}$	[12] [13] [14] [23] [24]	$\theta\{\alpha\}\{\beta\}$	219.97	209	10

			df
H_1: $\rho=0$	$G^2_{(6)}-G^2_{(1)}$	= 115.21	1
H_2: $\alpha_1=\ldots=\alpha_g=0$	$G^2_{(5)}-G^2_{(1)}$	= 49.91	15
H_3: $\beta_1=\ldots=\beta_g=0$	$G^2_{(4)}-G^2_{(1)}$	= 58.37	15
H_4: $\rho_{1(1)}=\ldots=\rho_{1(g)}=0$	$G^2_{(1)}-G^2_{(3)}$	= 2.76	15

6. Models for Relationships Among Actors with Nodal Attributes

Quite often, a researcher has information on the actors in a social group. Such information consists of individual measurements on the actors and is collected in addition to any sociometric relationships that exist between actors. These measurements are nodal attributes, and have been treated in past research as exogenous variables since no methods existed for incorporating them into a network study. In our introductory remarks to this chapter, we labelled this type of digraph set as type 3, univariate digraphs with data on nodal properties or attributes. In this section, we discuss several models for such data. We fit these models to the Galaskiewicz data in the next section.

Nodal attribute data can be recorded in a $(g \times p)$ data matrix, which we denote by $\underset{\sim}{A}$. Each row corresponds to one of the g actors, and each column to one of p variables. These variables may be either categorical or numerical, but for our purposes a numerical variable must be categorized into distinct, non-overlapping categories. For example, Larntz and Weisberg (1976) study interactions between $g=6$ new U.S. Navy recruits, and consider $p=2$ variables, the race of each recruit (Black and White) and the bunk is which each recruit sleeps (there were 3 adjacent two-tier bunks). Thus $\underset{\sim}{A}$ is a 6×2 array, with one categorical and one numerical discrete variable.

For a given $\underset{\sim}{A}$ array, we can group together all actors who have identical scores across the p variables. By permuting rows in $\underset{\sim}{A}$, such that the first g_1 rows $(a_1, a_2, \ldots a_{g_1})$ are identical, the next g_2 rows $(a_{g_1+1}, a_{g_1+2}, \ldots, a_{g_1+g_2})$ are identical, etc., we partition the full group of actors of size g into K subgroups of sizes $g_1, g_2, \ldots, g_K,$ such that the members of a given subgroup are identical with respect to the p variables

The Larntz and Weisberg example is shown in Table 10. Here, the partition of the six recruits on the two variables yields three groups, each containing two recruits.

Since we are treating the p nodal attributes as being categorical, we can use them to structure a p-dimensional cross-classified table, where we place the g actors into the appropriate cell in this table, keeping a roster of which actors are in a given cell. Some of these cells may have zero frequencies. There will be K cells that have non-zero frequencies and the set of actors common to a specific cell constitute a unique subgroup of actors, equivalent to one of those created by permuting the rows of the $\underset{\sim}{A}$ matrix and gathering together actors which have identical row scores across variables.

We now present two types of models for analyzing the sociometric relationships between the actors, classified into K subgroups. The first model is an extension of p_1 where we equate the $\{\alpha_i\}$ and $\{\beta_j\}$ parameters for actors within a given subgroup. The second model incorporates a set of parameters $\{\theta_{ij}\}$ for inter-and intra-group choices.

Note that the partition of the actors is accomplished by utilizing exogenous information--data gathered in addition to the relational data.

Table 10.

Larntz and Weisberg (1976) groupings for 6
U.S. Navy recruits

	Variable		
Recruit	Race	Bunk	Group
1	Black	1	G_1
2	Black	1	
3	White	2	G_2
4	White	2	
5	White	3	G_3
6	White	3	

This is fundamentally different than standard clustering algorithms which find subgroups or cliques solely from the relational data. For example, CONCOR, the basic algorithm for obtaining a blockmodel for a group of actors (Breiger, Boorman, and Arabie, 1975, and White, Boorman, and Brieger, 1976) uses only the sociomatrices to find hidden clique structures. The subgroups modelled with our methods are postulated to exist based on substantive evidence. In the absence of exogenous information, a researcher could find subgroups among the actors using a clustering algorithm, and then model the relationships between actors using the models discussed in this chapter. The similarities between our statistical approach to these problems and the qualitative clustering methods currently in vogue are intriguing.

6.1 A version of p_1 for subgroups

Suppose the group of actors, G, has been partitioned into K subgroups $G_1, G_2, \ldots, G_K$, such that subgroup G_m contains g_m actors with $\sum_{m=1}^{K} g_m = g$, and that the sociomatrix $\underset{\sim}{X}$ has been rearranged so that the first g_1 rows and columns correspond to the actors in G_1, the next g_2 rows and columns to those in G_2, etc.

First suppose that there are $K=2$ subgroups. Note that there are two kinds of choices, intragroup choices (actors in G_m choosing other actors in $G_m, m=1,2$) and intergroup choices (actors in G_m choosing actors in $G_n, m \neq n$). We postulate the following model for the logarithms of the probabilities for intragroup and intergroup choices:

$$
(18)\qquad
\begin{aligned}
\mu_{ij00} &= \lambda^{(mm)}\,, \\
\mu_{ij10} &= \lambda^{(mm)} + \alpha_m + \beta_m + \theta\,, \\
\mu_{ij01} &= \lambda^{(mm)} + \alpha_m + \beta_m + \theta\,, \\
\mu_{ij11} &= \lambda^{(mm)} + 2\alpha_m + 2\beta_m + 2\theta + \rho\,,
\end{aligned}
\qquad \text{if } i \text{ and } j \in G_m,\ m=1,2;
$$

$$
(19)\qquad
\begin{aligned}
\mu_{ij00} &= \lambda^{(mn)} \\
\mu_{ij10} &= \lambda^{(mn)} + \alpha_m + \beta_n + \theta\,, \\
\mu_{ij01} &= \lambda^{(mn)} + \alpha_n + \beta_m + \theta\,, \\
\mu_{ij11} &= \lambda^{(mn)} + \alpha_m + \alpha_n + \beta_m + \beta_n + 2\theta + \rho
\end{aligned}
\qquad
\begin{aligned}
&i \in G_m, \\
&j \in G_n,\ m \neq n;
\end{aligned}
$$

$$(20)\qquad \lambda^{(12)} = \lambda^{(21)}$$

and

$$(21)\qquad \alpha_1 + \alpha_2 = \beta_1 + \beta_2 = 0\,,$$

where the normalization constants $\lambda^{(mm)}$ and $\lambda^{(mn)}$ are chosen to ensure that the sets of four probabilities add to 1. This model is essentially a version of p_1, in which $\alpha_i = \alpha_{i'}$ and $\beta_i = \beta_{i'}$ if actors i and i' are in the same subgroup. The reciprocity parameters are equal for both intergroup and intragroup mutual relations.

Extending equations (18)-(21) to $K>2$ subgroups is straightforward. Equations (18) for intragroup choices apply for pairs of actors in the same group G_m, $m = 1,2,\ldots,K$, and equations (19) for intergroup choices for actors in different groups, G_m and G_n, $m,n=1,2,\ldots,K$, $m\neq n$. In addition,

$$(22)\qquad \lambda^{(mn)} = \lambda^{(nm)}$$

and

$$(23) \qquad \sum_{i=m}^{K} \alpha_m = \sum_{n=1}^{K} \beta_n = 0 .$$

All actors in subgroup G_m have a common α value, α_m, and a common β value, β_m. We label this model p_{K1}. It is equivalent to p_1 but applied to K subgroups rather than individuals.

In order to succinctly describe the computation of maximum likelihood estimates for this model we introduce some additional notation. Let

$$(24) \qquad w_{mnk\ell} = \sum_{i \varepsilon G_m} \sum_{j \varepsilon G_n} y_{ijk\ell}$$

be the sum of quantities $y_{ijk\ell}$ over all dyads such that the first actor is in subgroup G_m and the second is in subgroup G_n. These w's are the elements of $K \times K \times 2 \times 2$ dimensional array given in Table 11. Note that the entries in this table are not binary, since $w_{mn00} + w_{mn01} + w_{mn10} + w_{mn11} = g_m(g_n - \delta_{mn})$ where δ_{mn} is the Kronecker delta function,

$$(25) \qquad \delta_{mn} = \begin{cases} 1, & \text{if } m=n \\ 0, & \text{otherwise} . \end{cases}$$

The elements of $\underset{\sim}{w}$ have the same symmetries as the elements of $\underset{\sim}{y}$:

$$(26) \qquad \begin{aligned} w_{mn00} &= w_{nm00} , \text{ for all m and n ,} \\ w_{mn01} &= w_{nm10} , \text{ for } m \neq n , \\ w_{mm01} &= w_{mm10} , \text{ for all m ,} \\ w_{mn11} &= w_{nm11} , \text{ for all m and n .} \end{aligned}$$

Table 11

The $\underset{\sim}{w}$ array for model p_{K1}, the extension of p_1 to K subgroups

	G_1		G_2		.	.	.	G_K	
G_1	w_{1100}	w_{1101}	w_{1200}	w_{1201}	.	.	.	w_{1K00}	w_{1K01}
	w_{1110}	w_{1111}	w_{1210}	w_{1211}				w_{1K10}	w_{1K11}
G_2	w_{2100}	w_{2101}	w_{2200}	w_{2201}				w_{2K00}	w_{2K01}
	w_{2100}	w_{2111}	w_{2210}	w_{2211}				w_{2K10}	w_{2K11}
	.		.					.	
	.		.		.	.	.	.	
	.		.					.	
G_K	w_{K100}	w_{K101}	w_{K200}	w_{K201}	.	.	.	w_{KK00}	w_{KK01}
	w_{K110}	w_{K111}	w_{K210}	w_{K211}				w_{KK10}	w_{KK11}

$$w_{mnk\ell} = \sum_{i\varepsilon G_m} \sum_{j\varepsilon G_n} y_{ijk\ell} \ .$$

The log likelihood function associated with this model can be written as

$$\log L(\{\lambda^{(mn)}\}, \theta, \{\alpha_m\}, \{\beta_n\}, \rho|\underset{\sim}{w})$$

(27)
$$= \theta w_{++1+} + \sum_{m=1}^{K} \alpha_m w_{m+1+} + \sum_{n=1}^{K} \beta_n w_{+n1+} + \frac{\rho}{2} w_{++11} + \sum_{m\leq n} \lambda^{(mn)} w_{mn++}$$

Because of the symmetries (26), the minimal sufficient statistics for the density function p_{K1} are:

(28)
$$\begin{array}{ll} \lambda: & (w_{mn++}) \\ \rho: & (w_{++11}) \\ \{\alpha_m\}: & (w_{m+1+}) \ , \ (w_{+m+1}) \\ \{\beta_n\}: & (w_{+n1+}) \ , \ (w_{n++1}) \ . \end{array}$$

These quantities are the six two-way margins of the $\underset{\sim}{w}$-array, and from the general results on loglinear models the maximum likelihood estimates of the parameters are found by setting them equal to their expected values.

Note that the $\underset{\sim}{w}$-table is an aggregated version of the original $\underset{\sim}{y}$-table and thus we must introduce some additional notation for the expected value for the (m,n,k,ℓ) entry in $\underset{\sim}{w}$. Thus we let

(29)
$$m^*_{mnk\ell} = \sum_{i\varepsilon G_m} \sum_{j\varepsilon G_n} m_{ijk\ell} = E(w_{mnk\ell}) \ .$$

Then, it can be shown that fitting the p_{k1} model to a sociomatrix is equivalent to "fitting" the following marginal totals from the $\underset{\sim}{w}$-table to $\underset{\sim}{\hat{m}}^*$:

(30) [12] [13] [14] [23] [24] [34] .

The loglinear model being fitted to the w-table is not, however, the no three-factor interaction model, as was the case for p_1. The reason for this is as follows.

Note that the entries in the $\underset{\sim}{w}$-array that count the number of intragroup null and mutual relationships are double the actual number of such relationships. These doubled frequencies occur in the w_{mm00} and w_{mm11} cells, m=1,2,...,K. Thus, to get a table of expected values, $(m^*_{mnk\ell})$, corresponding to $\underset{\sim}{w}$, we must take as initial values for the standard iterative proportional scaling algorithm not a matrix of 1's but rather a matrix with entries

$$\text{(31)} \qquad \hat{m}^{*(0)}_{mnk\ell} = 1 + \delta_{mn}\,\delta_{k\ell} \quad ,$$

where δ_{mn} is defined in equation (25). The cells with doubled frequencies have initial values of 2, while all other cells have initial values of 1. As a consequence, the initial values have a specified 4-factor interaction structure corresponding to doubled frequencies, and iteratively adjusting these initial values for the six sets of two-way marginal totals will preserve this interaction structure. Most multidimensional contingency table programs that utilize the method of iterative scaling allow the user to specify starting values, so this is not a serious complication.

A formal proof of the equivalence of fitting p_{K1} to $\underset{\sim}{X}$ using the generalized iterative scaling algorithm of Darroch and Ratcliff (1972), and fitting the loglinear model with adjusted starting values to the $\underset{\sim}{w}$-array as outlined above, involves extensions of the proof for p_1 mentioned earlier, and a formulation of loglinear models for tables with duplicated and doubled entries, described in Fienberg and Wasserman (1979) for a closely related problem.

Fitting the two-way margins in (30) with the initial values in (31) yields estimated expected values, $(\hat{m}^*_{mnk\ell})$ for the $\underset{\sim}{w}$-array which itself was an aggregated version of the original table. To get the estimated expected values for the original $\underset{\sim}{y}$-array, we simply divide the $(\hat{m}^*_{mnk\ell})$ by the number of dyads that have been aggregated:

$$(32) \qquad \hat{m}_{ijk\ell} = \frac{1}{g_m(g_m-1)}\,\hat{m}^*_{mmk\ell} \qquad i,j \in G_m,\ m=1,2,\ldots,K\ ,$$

and

$$(33) \qquad \hat{m}_{ijk\ell} = \frac{1}{g_m g_n}\,\hat{m}^*_{mnk\ell} \qquad i \in G_m,\ j \in G_n\ ,\ m,n=1,2,\ldots,K\ ,\ m \neq n\ .$$

Note that this disaggregation reflects the original doubling of the frequencies in the sums for the mutual and null intragroup relationships, w_{mm00} and w_{mm11} , and the counting of all asymmetric intragroup relationships in w_{mm10} and w_{mm01} , which are thus equal. The latter implies that

$$(34) \qquad \hat{m}_{ij10} = \hat{m}_{ij01} \qquad \text{for} \qquad i,j \in G_m \quad m=1,2,\ldots,K\ .$$

For the P_{K1} model, we are estimating 2K parameters: K-1 for the $\{\alpha_k\}$, K-1 for the $\{\beta_k\}$, 1 for θ , and 1 for ρ . Thus there are $g(g-1)-2K$ degrees of freedom associated with the model. The likelihood ratio goodness-of-fit statistic for this model is

$$(35) \qquad G^2 = 2 \sum_{i<j} \sum_{k,\ell} y_{ijk\ell} \log \frac{y_{ijk\ell}}{\hat{m}_{ijk\ell}}$$

$$= -2 \sum_{i<j} \log \hat{m}_{ijk'\ell'} \quad \text{where} \quad X_{ij} = k' \text{ and } X_{ji} = \ell'$$

$$= -\left(2 \sum_{m \neq n} \sum_{k,\ell} w_{mnk\ell} \log \frac{\hat{m}^*_{mnk\ell}}{g_m g_n} + \sum_{m} \sum_{k,\ell} w_{mmk\ell} \log \frac{\hat{m}^*_{mmk\ell}}{g_m(g_m-1)}\right)$$

where the last equation for G^2 is expressed in terms of the observed and estimated expected entries in the aggregated $\underset{\sim}{w}$-array.

6.2 A subgroup model with group choice parameters

The previous model for subgroups, p_{K1}, is a version of p_1 in which all actors in a subgroup have a common α and β. Thus α_m is a measure of the expansiveness of subgroup G_m and β_n is a measure of the attractiveness of G_n. The sum

$$\theta_{mn} = \alpha_m + \beta_n + \theta \tag{36}$$

measures how likely it is that an actor in G_m chooses an actor in G_n. Here we do not necessarily assume a linear decomposition of the $\{\theta_{mn}\}$ and we also allow for differential rates of reciprocated arcs forming both within and between groups. We call the resulting model $p_{K.5}$ to distinguish it from its competitor, p_{K1}.

Suppose that we have K subgroups, $G_1, G_2, \ldots, G_K$, of sizes $g_1, g_2, \ldots, g_K$, $\Sigma g_m = g$. Let the logarithms of the probabilities of the four dyad states be:

$$\begin{aligned}
\mu_{ij00} &= \lambda^{(mn)} \\
\mu_{ij10} &= \lambda^{(mn)} + \theta_{mn} \\
\mu_{ij01} &= \lambda^{(mn)} + \theta_{nm} \\
\mu_{ij11} &= \lambda^{(mn)} + \theta_{mn} + \theta_{nm} + \rho_{mn}
\end{aligned} \tag{37}$$

where $i \varepsilon G_m$ and $j \varepsilon G_n$, and for $m \neq n$,

$$\lambda^{(mn)} = \lambda^{(nm)} \tag{38}$$

$$\rho_{mn} = \rho_{nm} .$$

There are no constraints on the θ's and ρ's , and the $\lambda^{(mn)}$ are included, as usual, for normalization purposes, i.e., to make probabilities add to 1. Note that $p_{K.5}$ would be identical to p_{K1} if we were to allow a linear decomposition of the θ's , as in equation (36), and then equate all the ρ's.

As with the other models discussed in this chapter, we assume that the dyads are independent. The log-likelihood function for $p_{K.5}$ has two components, one for intragroup choices, and another for intergroup choices:

$$\log L(\{\lambda^{(mn)}\} , \{\theta_{mn}\} , \{\rho_{mn}\} | \underset{\sim}{y})$$
$$= \sum_{m=1}^{K} \left[\frac{g_m(g_m-1)}{2} \lambda^{(mm)} + \theta_{mm} w_{mm1+} + \frac{\rho_{mm}}{2} w_{mm11} \right]$$
$$+ \sum_{\substack{m,n \\ m<n}}^{K} \left[g_m g_n \lambda^{(mn)} + \theta_{mn} w_{mn1+} + \theta_{nm} w_{mn+1} + \rho_{mn} w_{mn11} \right]$$

The $(w_{mnk\ell})$ are defined in equation (24). There are K intragroup choice components in the log likelihood, and $\binom{K}{2}$ intergroup choice components. Note that these $K + \binom{K}{2}$ components have no parameters in common, and can be considered separately.

The minimal sufficient statistics for the m^{th} intragroup choice component are w_{mm1+} , $\frac{1}{2} w_{mm11}$, and for the $(m,n)^{th}$ intergroup choice component are w_{mn1+}, w_{mn+1}, w_{mn11} . Since $w_{mm+1} = w_{mm1+}$, we can re-express the sufficient statistics as the entries in the w-array. Once again from the general results on loglinear models, the maximum likelihood estimates of the parameters are found by equating them to their expected values.

Thus, defining $m^*_{mnk\ell}$ as the expected value for the $(m,n,k\ \ell)$th entry of $\underset{\sim}{w}$ (equation 29), we observe that

$$\hat{m}^*_{mnk\ell} = w_{mnk\ell} \tag{40}$$

for the model $p_{K.5}$. We need not iterate to find estimates of the expected values of the $\underset{\sim}{w}$-array. For network data without nodal attribut... labelled this model $p_{.5}$; thus, we call the extension of this mo... K subgroups, $p_{K.5}$.

The estimated expected values $(\hat{m}^*_{mnk\ell})$, defined in equatio... (40... the $\underset{\sim}{w}$-array, again are an aggregated version of the original sociomat... estimated expected values for the $\underset{\sim}{y}$-array, are found by dividi... the $(\hat{m}^*_{mnk\ell})$ be the number of dyads that have been aggregated, as in equ... (32) and (33).

The degrees of freedom for $p_{K.5}$ are slightly more complicated to compute than for p_{K1}. In the K diagonal 2×2 matrices in the $\underset{\sim}{w}$-array, there are 2K parameters, K θ's and K ρ's. In the $\binom{K}{2}$ off-diagonal 2×2 matrices, there are $3\binom{K}{2}$ parameters, 2 θ's and 1 ρ for every matrix. Thus, $p_{K.5}$ has $g(g-1)-2K-3\binom{K}{2} = g(g-1) - \frac{K}{2}(3K+1)$ degrees of freedom. The likelihood ratio statistic is computed as in equation (35), substituting $(w_{mnk\ell})$ for the $(\hat{m}^*_{mnk\ell})$.

7. The Example Continued

We now apply the models discussed in the previous section to the 73 organizations in Towertown. There are three sociometric relations defined between pairs of organizations: information (denoted by "I"), money ("M"), and support ("S").

We consider two variables to partition the 73 actors: (1) Whether each organization is owned by people in the community ("local") or by people outside the community ("extralocal"); and (2) Whether each organization has public or private ownership. Both of these variables are dichotomous so that there are $K = 2^2 = 4$ subgroups. We list, in Table 12, the organizations in each of the 4 subgroups. There are, of course, many other ways to partition 73 organizations. The local/extralocal × public/private split is simple enough to illustrate our methods and is of substantive interest. A complete analysis of the relationships among the organizations would involve a study of the flows within alternative and perhaps more elaborate partitions.

We do not include the three 73×73 sociomatrices here because of their size, but we do report aggregate versions of them in the form of the $\underset{\sim}{w}$-arrays for our four subgroups, in Table 13. We give only the upper triangle of these arrays because of the symmetries shown in equation (26). The $p_{4,1}$ and $p_{4,.5}$ densities were fitted to each of these arrays, yielding the following values of the likelihood ratio goodness-of-fit statistics:

	$p_{4,1}$		$p_{4,.5}$	
	G^2	df	G^2	df
Information	4592.4	5248	4415.4	5220
Money	3044.7	5248	2941.0	3811
Support	4062.0	5248	3965.2	5225

Table 12

4 subgroups of the 73 organizations in Towertown

G_1 Private Local	G_2 Private Extralocal	G_3 Public Local	G_4 Public Extralocal
Clothing Mfg. Co.	Farm Bureau	City Council	Highway Authority
Farm Supply Co.	Farm Equipment Co.	City Manager	State University
Chamber of Commerce	Mechanical Co.	County Board	Dept. of Public Aid
Banker's Association	Electric Equip. Co.	Fire Department	Employment Services
1st Towertown Bank	Metal Products Co.	Human Relations Committee	(g_4=4)
Towertown Svgs.&Loan	Music Equipment Co.	Mayor's Office	
Bank of Towertown	Music Emp. Union #1	Police Department	
2nd Towertown Bank	Music Emp. Union #2	Sanitary District	
Brinkman Law Firm	Teachers' Union	Streets and Sanitation	
Cater Law Firm	League of Women Voters	Park District	
Lenhart Law Firm	1st Kiwanis Club	Zoning Board	
Bar Association	2nd Kiwanis Club	Hospital Board	
Board of Realtors	Rotary Club	Public Hospital	
Small Business Assoc.	Lions Club	Board of Mental Health	
Central Labor Union	Parent-Teacher Assoc.	County Board of Health	
Democratic Committee	St. Hilary's Catholic Church	School Board	
Republican Committee	1st Baptist Church	High School	
Towertown News	1st Church of the Light	Local Community College	
WTWR Radio	1st Congregational Church	Housing Authority	
Medical Society	1st Methodist Church	Towertown Mental Health Center	
Health Services Center	Unity Lutheran Church	Youth Services Bureau	
United Fund	University Methodist Church	(g_3=21)	
1st Assoc. of Churches	(g_2=22)		
2nd Assoc. of Churches			
Family Services			
YMCA			
(g_1=26)			

Table 13

$\underset{\sim}{w}$-arrays for intra- and inter-group choices among the 4 subgroups

Information:

	G_1		G_2		G_3		G_4	
G_1	404	44	422	20	365	33	68	4
	44	158	30	100	43	105	2	30
G_2			382	24	356	29	74	3
			24	32	20	57	1	10
G_3					236	33	49	7
					33	118	1	27
G_4							8	0
							0	4

Money:

	G_1		G_2		G_3		G_4	
G_1	424	81	433	97	459	43	88	8
	81	64	24	18	30	14	4	4
G_2			454	4	435	0	83	0
			4	0	27	0	5	0
G_3					348	30	74	8
					30	12	2	0
G_4							10	1
							1	0

Support:

	G_1		G_2		G_3		G_4	
G_1	484	56	463	44	420	17	80	1
	56	54	34	31	61	48	13	10
G_2			402	26	361	6	73	1
			26	8	60	35	12	2
G_3					254	40	55	8
					40	86	8	13
G_4							6	3
							3	0

The values of G^2 are in all cases less than the degrees of freedom. Note that the degrees of freedom for $p_{4,.5}$ vary from one generator to another. This is the case because we have adjusted them for the zero counts in the $\underset{\sim}{w}$-arrays (see the discussion of this point in Bishop, Fienberg, and Holland, 1975, pp. 115-116, and in Fienberg, 1977, p. 109-110). These zeros constrain the corresponding $\hat{m}_{ijk\ell}$ to be equal to zero, and this must be taken into account in the computation.

While both the $p_{4,1}$ and $p_{4,.5}$ models seem to provide an adequate fit to the data, whether they are suitable simplications of p_1 or $p_{.5}$ can be determined only by a direct comparison of $p_{4,1}$ with p_1 and of $p_{4,.5}$ with $p_{.5}$. Unfortunately we have been unable to make such a comparison to date, because we have been unable to fit p_1 and $p_{.5}$ to the 73×73 sociomatrices due to storage limitations on our computer. We hope to recitfy this situation in the near future, and for the moment our assessment of the fit of these models is qualitative in nature.

In Table 14 we give the estimated dyadic probabilities for the $p_{4,1}$ model. Those for the $p_{4,.5}$ model can be calculated directly from Table 13, by normalizing each 2×2 table so that its entries add to 1. For flows of information and support, the estimated dyad probabilities are quite similar except for the null relationships within G_3, and between G_2 and G_4. There is considerable reciprocity, with within and between subgroups except for within G_2, with reciprocity of support being much less than for information. For money, on the other hand, the estimated dyadic probabilities differ primarily between G_1 and G_2. Actually comparisons between the two models for choices among and between G_2, G_3, and G_4 are difficult to make because of all of the zero counts in Table 13. All in all, the $p_{4,1}$ model seems

Table 14

Normalized fitted values from $p_{4,1}$ for the three generators

Information:

	G_1		G_2		G_3		G_4	
G_1	0.671	0.038	0.739	0.050	0.623	0.064	0.671	0.085
	0.038	0.253	0.072	0.140	0.079	0.234	0.059	0.238
G_2			0.868	0.025	0.741	0.065	0.731	0.086
			0.025	0.082	0.055	0.140	0.041	0.142
G_3					0.672	0.038	0.616	0.094
					0.038	0.251	0.053	0.237
G_4							0.662	0.039
							0.039	0.261

Money:

	G_1		G_2		G_3		G_4	
G_1	0.693	0.090	0.793	0.160	0.779	0.104	0.810	0.113
	0.090	0.126	0.130	0.016	0.087	0.031	0.056	0.021
G_2			0.969	0.014	0.901	0.017	0.929	0.019
			0.014	0.002	0.077	0.004	0.050	0.003
G_3					0.939	0.027	0.909	0.054
					0.027	0.008	0.032	0.005
G_4							0.961	0.019
							0.019	0.004

Support:

	G_1		G_2		G_3		G_4	
G_1	0.826	0.044	0.816	0.080	0.689	0.056	0.730	0.043
	0.044	0.086	0.080	0.049	0.147	0.109	0.147	0.080
G_2			0.912	0.028	0.747	0.038	0.777	0.029
			0.028	0.032	0.147	0.068	0.145	0.049
G_3					0.715	0.058	0.696	0.082
					0.058	0.170	0.106	0.115
G_4							0.815	0.046
							0.046	0.094

quite reasonable, and $p_{4.5}$ appears to offer few substantial gains over it.

Next we examine the maximum likelihood estimates for the parameters in the $p_{4,1}$ model. These are computed as in Section 3.3, taking ratios of ratios of the elements of the arrays in Table 14, and are given in Table 15.

We see immediately that information flows are reciprocated more often than support, and support flows, as noted earlier, much more often than money. The $\{\hat{\alpha}_i\}$ parameters are not very different for information and money, except for $\hat{\alpha}_1$--G_1 is more likely to send money to the other subgroups than information (and also support). The least likely outflow is money from G_2. There is also a small tendency for money and support not to flow from G_4. The remaining $\{\hat{\alpha}_i\}$ are smaller in magnitude and of little importance.

The $\{\hat{\beta}_i\}$ "attractiveness" parameter estimates differ as much across groups as the $\{\hat{\alpha}_i\}$ parameters, and on average, are slightly larger. We note that support is likely to flow to subgroups G_3 and G_4, information is likely to flow to subgroup G_4, and money to G_1. Large negative parameters indicate that neither support nor information is not likely to flow to G_2.

In conclusions, we note that reciprocal flows of money come to and go from G_1. Subgroup G_2 is unlikely to send money, but does receive information and support. Subgroup G_3 receives only support, while G_4 receives support and information, but is unlikely to send support and money. By studying the substantive nature of the subgroups, we see that public organizations are likely to receive support, which is more likely to come from the private organizations. Extralocal organizations are likely to receive information, but very unlikely to send money. The only strong reciprocated flows are of money, to and from the private local organizations.

Table 15

Parameter Estimates for the Three Relations

	Information	Money	Support
$\hat{\alpha}_1$	0.057	1.028	0.237
$\hat{\alpha}_2$	-0.103	-0.899	0.155
$\hat{\alpha}_3$	0.158	0.171	-0.041
$\hat{\alpha}_4$	-0.112	-0.300	-0.351
$\hat{\beta}_1$	0.110	0.388	0.217
$\hat{\beta}_2$	-0.418	0.135	-0.681
$\hat{\beta}_3$	0.005	-0.284	0.467
$\hat{\beta}_5$	0.302	-0.239	0.422
$\hat{\theta}$	-2.340	-2.761	-2.260
$\hat{\rho}$	3.369	0.982	2.217

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